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Field Theoretic Cogalois Theory via Abstract Cogalois Theory

Toma Albu*

Koç University, Department of Mathematics, Rumeli Feneri Yolu, 34450 Sariyer - Istanbul, Turkey
Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1 - 764, RO - 014700 Bucharest 1, Romania

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Abstract

Abstract Cogalois Theory for arbitrary profinite groups was initiated by T. Albu and Ş.A. Basarab [An Abstract Cogalois Theory for profinite groups, J. Pure. Appl. Algebra 200 (2005) 227–250]. The aim of this paper is twofold: firstly, to present the abstract group theoretic versions of various types of Kummer field extensions, and secondly, to show how some basic results of the (Field Theoretic) Cogalois Theory can be very easily deduced from their abstract versions.

MSC: 20E18; 12G05; 12F10; 12F99

Introduction

This paper is a natural continuation of [4] were an abstract group theoretic framework of Cogalois Theory has been developed for arbitrary profinite groups. *Cogalois Theory* (see [3]) is a fairly recent domain of Field Theory investigating field extensions, finite or not, which possess a Cogalois correspondence. This theory is somewhat dual to the very classical *Galois Theory* dealing with field extensions possessing a Galois correspondence.

In our paper [4] there was no indication about how the field theoretic results of Cogalois Theory can be deduced from their abstract group theoretic correspondents. This is done in the present paper. Besides that, the abstract group theoretic versions of various types of Kummer field extensions are introduced and investigated.

The paper is divided into four sections. In Section 0 we introduce the concepts of *Cogalois group* and *strongly Cogalois group* of an arbitrary profinite group Γ acting continuously on a discrete subgroup A of the abelian group $\mathbb{Q}/\mathbb{Z}$, and characterize strongly Cogalois groups among the radical groups of Γ.

In Section 1 we introduce four types of *Kummer groups of cocycles*; these are precisely the abstract group theoretic correspondents of the various types of Kummer field extensions studied in Galois Theory and Cogalois Theory. We prove that all of them are Cogalois groups of cocycles.

Section 2 presents a dictionary relating the basic notions of the (Field Theoretic) Cogalois Theory like: $\mathbb{G}$-*radical extension*, $\mathbb{G}$-*Kneser extension*, $\mathbb{G}$-*Cogalois extension*, and *Cogalois extension*, to their correspondents in the (Group

* Corresponding address: Koç University, Department of Mathematics, Rumeli Feneri Yolu, 34450 Sariyer - Istanbul, Turkey.
E-mail addresses: talbu@ku.edu.tr, Toma.Albu@imar.ro.

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Theoretic) Abstract Cogalois Theory: *G-radical group*, *G-Kneser group*, *G-Cogalois group*, and *strongly Cogalois group*, respectively. This dictionary will allow us to easily retrieve in the next section some of the basic facts of the (Field Theoretic) Cogalois Theory like: the *Kneser Criterion*, the *General Purity Criterion*, and the uniqueness of the Kneser group of a $\mathbb{G}$-Cogalois field extension, from their abstract versions.

0. Preliminaries

Throughout this section Γ will always denote a fixed profinite group, and A will be a fixed subgroup of the abelian group $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on A endowed with the discrete topology. For terminology and notation concerning Abstract Cogalois Theory the reader is referred to [4].

The abstract correspondents for subgroups of $Z^1(\Gamma, A)$ of Kneser and Cogalois field extensions have been introduced in [4]. If one moves now via the maps $(-)^{\perp}$ from subgroups of $Z^1(\Gamma, A)$ to subgroups of the given profinite group Γ, then one can define the abstract versions for the latter of the concepts of radical, simple radical, Kneser, and Cogalois field extension. As in [6], a subgroup Δ of Γ is said to be *G-radical* if $\Delta = G^{\perp}$ for some $G \leqslant Z^1(\Gamma, A)$, and *G-Kneser* if Δ is G-radical and G is a Kneser group of $Z^1(\Gamma, A)$. A *radical group* (resp. a *Kneser group*) of Γ is a subgroup which is G-radical (resp. G-Kneser) for some $G \leqslant Z^1(\Gamma, A)$.

Definition 0.1. A subgroup Δ of Γ is said to be *Cogalois* (resp. *simple radical*) if there exists a Cogalois group G of $Z^1(\Gamma, A)$ (resp. $g \in Z^1(\Gamma, A)$) such that $\Delta = G^{\perp}$ (resp. $\Delta = g^{\perp}$). We call Δ *strongly Cogalois* if $\Delta = \Delta^{\perp\perp}$ and $\Delta^{\perp}$ is a Cogalois group of $Z^1(\Gamma, A)$.

If Δ is Cogalois, then the Cogalois group G of $Z^1(\Gamma, A)$ for which $\Delta = G^{\perp}$ is uniquely determined by [4, Corollary 2.12], and we say in this case that Δ is *G-Cogalois*.

Lemma 0.2. *The following statements are equivalent for a radical subgroup Δ of Γ.*

(1) *Δ is strongly Cogalois.*
(2) *$\Delta^{\perp}$ is a Kneser group of $Z^1(\Gamma, A)$.*

Proof. (2) $\Longrightarrow$ (1): Since Δ is a radical subgroup of Γ, there exists $H \leqslant Z^1(\Gamma, A)$ such that $\Delta = H^{\perp}$, and so $\Delta^{\perp\perp} = (H^{\perp})^{\perp\perp} = H^{\perp} = \Delta$. If $G := \Delta^{\perp}$ is not a Cogalois group of $Z^1(\Gamma, A)$, then by the Quasi-Purity Criterion [4, Theorem 2.5], there exists $p \in \mathcal{P}(\Gamma, A) \cap \mathcal{P}_G$ such that $\Delta = G^{\perp} \leqslant \varepsilon_p^{\perp}$, and hence $\varepsilon_p^{\perp\perp} \leqslant \Delta^{\perp} = G$. Consequently, $\varepsilon_p \in \varepsilon_p^{\perp\perp} \leqslant G$ if $p \neq 4$, and $\varepsilon_4' \in \varepsilon_4^{\perp\perp} \leqslant G$ if $p = 4$. By the Abstract Kneser Criterion [4, Theorem 1.20] we deduce that G is not Kneser.

(1) $\Longrightarrow$ (2): This is trivial since any Cogalois group of $Z^1(\Gamma, A)$ is also Kneser. □

1. Kummer groups of cocycles

In this section we introduce four types of *Kummer groups of cocycles*, which are the abstract group theoretic correspondents of the various types of Kummer field extensions studied in Galois Theory and Cogalois Theory (see [3]), and prove that each of them is a Cogalois group of cocycles.

Definition 1.1. Let $G \leqslant Z^1(\Gamma, A)$, and let $n \in \mathbb{N}$.

(1) G is said to be a *classical n-Kummer group* if $nG = \{0\}$ and $A[n] \subseteq A^{\Gamma}$.
(2) G is said to be a *generalized n-Kummer group* if $nG = \{0\}$ and $A^{G^{\perp}}[n] \subseteq A^{\Gamma}$.
(3) G is said to be an *n-Kummer group with few cocycles* if $nG = \{0\}$ and $A^{G^{\perp}}[n] \subseteq A[2]$.
(4) G is said to be an *n-quasi-Kummer group* if $nG = \{0\}$ and $A[p] \subseteq A^{\Gamma}$ for every $p \in \mathcal{P}_n$.

We say that G is a *classical Kummer group* (resp. a *generalized Kummer group, Kummer group with few cocycles, quasi-Kummer group*) if G is a classical m-Kummer group (resp. a generalized m-Kummer group, m-Kummer group with few cocycles, m-quasi-Kummer group) for some $m \in \mathbb{N}$.

Since $A[2] \subseteq \{\widehat{0}, \widehat{1/2}\} \subseteq A^{\Gamma}$, any n-Kummer group with few cocycles is a generalized n-Kummer group. Clearly, any classical n-Kummer group is both a generalized n-Kummer group and an n-quasi-Kummer group.

Proposition 1.2. *Any generalized Kummer group and any quasi-Kummer group is Cogalois. In particular, any classical Kummer group and any Kummer group with few cocycles is Cogalois.*

Proof. Let $G \leqslant Z^1(\Gamma, A)$. If G is a generalized Kummer group, then $nG = \{0\}$ and $A^{G^\perp}[n] \subseteq A^\Gamma$ for some $n \in \mathbb{N}$. If $p \in \mathcal{P}_G$, then $p \mid n$, and hence $A^{G^\perp}[p] \subseteq A^{G^\perp}[n] \subseteq A^\Gamma$, i.e., the subgroup A^Γ of $A^{G^\perp}$ is quasi-$\mathcal{P}_G$-pure. By [4, Theorem 2.5] we deduce that G is a Cogalois group of $Z^1(\Gamma, A)$.

If G is a quasi-Kummer group, then $\exists n \in \mathbb{N}$ with $nG = \{0\}$ and $A[p] \subseteq A^\Gamma, \forall p \in \mathcal{P}_n$. Since $\mathcal{P}_G \subseteq \mathcal{P}_n$, we have $A^{G^\perp}[p] \subseteq A[p] \subseteq A^\Gamma, \forall p \in \mathcal{P}_G$, i.e., the subgroup A^Γ of $A^{G^\perp}$ is quasi-$\mathcal{P}_G$-pure. Again by [4, Theorem 2.5] we deduce that G is a Cogalois group of $Z^1(\Gamma, A)$. $\square$

Corollary 1.3. *Let $G \leqslant Z^1(\Gamma, A)$ be one of the four types of Kummer groups of cocycles introduced in Definition 1.1. Then the maps*

$$(-)^\perp : \mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma | G^\perp) \quad \textit{and} \quad G \cap (-)^\perp : \overline{\mathbb{L}}(\Gamma | G^\perp) \longrightarrow \mathbb{L}(G)$$

are lattice anti-isomorphisms, inverse to one another.

Proof. By Proposition 1.2, G is a Cogalois group of $Z^1(\Gamma, A)$; by [4, Definition 2.1] we are done. $\square$

Proposition 1.4. *Let $G \leqslant Z^1(\Gamma, A)$ be a Cogalois group of bounded order such that $A[\exp(G)] \subseteq A^{G^\perp}$. Then G is a quasi-Kummer group.*

Proof. Set $n = \exp(G)$, and let $p \in \mathcal{P}_n$. Then $A[p] \subseteq A[n] \subseteq A^{G^\perp}$, and hence $A[p] \subseteq A^{G^\perp}[p]$. Since G is a Cogalois group of $Z^1(\Gamma, A)$, A^Γ is a quasi-$\mathcal{P}_G$-pure subgroup of $A^{G^\perp}$ by [4, Theorem 2.5]; hence $A^{G^\perp}[p] \subseteq A^\Gamma$ for all $p \in \mathcal{P}_G$. Since $n = \exp(G)$, we have $\mathcal{P}_G := \mathcal{O}_G \cap \mathcal{P} = \mathcal{P}_n$. Consequently, $A[p] \subseteq A^\Gamma$ for every $p \in \mathcal{P}_n$, which shows that G is an n-quasi-Kummer group. $\square$

Proposition 1.5. *Any generalized Kummer group $G \leqslant Z^1(\Gamma, A)$ with $A[\exp(G)] \subseteq A^{G^\perp}$ is a classical Kummer group.*

Proof. Assume that G is a generalized m-Kummer group, and let $n = \exp(G)$. Since $n \mid m$, we have $A[n] \subseteq A^{G^\perp}[n] \subseteq A^{G^\perp}[m] \subseteq A^\Gamma$ by hypothesis. Thus, G is a classical n-Kummer group. $\square$

Remarks 1.6. (1) The result in Proposition 1.4 is the abstract version of the following field theoretic result: *Any Galois n-bounded G-Cogalois extension E/F is an n-quasi-Kummer extension* [3, Theorem 13.4.3]. The condition $A[n] \subseteq A^{G^\perp}$, $n = \exp(G)$, in Proposition 1.4 corresponds to the fact that $\zeta_n \in E$, which, in turn, is a consequence of E/F being Galois. It is not clear whether we can replace it by the following one: *G is a Γ-submodule of $Z^1(\Gamma, A)$* (see also Corollary 2.2).

(2) The same question holds for Proposition 1.5, which is the abstract version of [3, Thm. 13.4.4]: *Any Galois generalized Kummer extension is a classical Kummer extension*.

2. A Field Theoretic $\longleftrightarrow$ Abstract Cogalois Theory dictionary

In this section we establish a dictionary, needed in the next section, relating the basic notions of the (Field Theoretic) Cogalois Theory to their correspondents in the Abstract Cogalois Theory.

Throughout this section Ω/F denotes a fixed Galois extension with the (profinite) Galois group $\Gamma := \mathrm{Gal}(\Omega/F)$. In particular, we can take as Ω an algebraic separable closure $\widetilde{F}^{\mathrm{sep}}$ of F. For basic notation, notions, and results of (Field Theoretic) Cogalois Theory the reader is referred to [3].

For any nonempty subset $S \subseteq \Omega$ we denote by $\mu(S)$ the set of all roots of unity contained in S, and for $n \in \mathbb{N}$, $\mu_n(S)$ will denote the set of all n-th roots of unity contained in S. If $\Omega = \widetilde{F}^{\mathrm{sep}}$ and $\mathrm{Char}(F) = p$, then the multiplicative group $\mu(\Omega)$ is isomorphic in a non-canonical way to the additive group $\mathbb{Q}/\mathbb{Z}$ if $p = 0$, and to its subgroup $\bigoplus_{q \in \mathbb{P} \setminus \{p\}} (\mathbb{Q}/\mathbb{Z})(q)$ for $p \neq 0$. Thus, the group $A := \mu(\Omega)$ is isomorphic to a uniquely determined subgroup of $\mathbb{Q}/\mathbb{Z}$, and the canonical action of Γ on Ω induces a continuous action of Γ on the discrete abelian torsion group A.

The exact sequence $\{1\} \longrightarrow A \longrightarrow \Omega^* \longrightarrow \Omega^*/A \longrightarrow \{1\}$ of topologically discrete Γ-modules yields the exact sequence of cohomology groups in low dimensions

$$\{1\} \longrightarrow A^{\Gamma} \longrightarrow \Omega^{*\Gamma} \longrightarrow (\Omega^*/A)^{\Gamma} \longrightarrow H^1(\Gamma, A) \longrightarrow H^1(\Gamma, \Omega^*).$$

Since $H^1(\Gamma, \Omega^*) = \{1\}$ by *Hilbert's Theorem 90*, we obtain a canonical epimorphism

$$\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A), x \mapsto (\sigma \in \Gamma \mapsto (\sigma x)x^{-1} \in A),$$

of abelian torsion groups, with $\mathrm{Ker}(\psi) = F^*$ and

$$T(\Omega/F) := \{x \in \Omega^* \mid (\sigma x)x^{-1} \in A, \forall \sigma \in \Gamma\} = \{x \in \Omega^* \mid \exists\, n \in \mathbb{N}, x^n \in F\}.$$

The quotient group $\mathrm{Cog}(E/F) := T(\Omega/F)/F^*$ is called in Cogalois Theory (see [3]) the *Cogalois group* of the field extension Ω/F. Note that the term "coGalois group" was also used in [7] with a completely different meaning, involving the concept of $\mathcal{F}$-cover of a module.

The epimorphism ψ induces a canonical group isomorphism

$$\varphi : \mathrm{Cog}(\Omega/F) \xrightarrow{\sim} Z^1(\Gamma, A)$$

(see also [2, Corollary 1.2], [3, Theorem 15.1.2], or [8, Remark, p.580]), which identifies the subgroups $G \leqslant Z^1(\Gamma, A)$ investigated in the framework of Abstract Cogalois Theory with the subgroups $\mathbb{G}/F^* := \varphi^{-1}(G) \leqslant \mathrm{Cog}(\Omega/F)$ investigated in the framework of (Field Theoretic) Cogalois Theory.

The lattice $\mathbb{I}(\Omega/F)$ of all intermediate fields of the extension Ω/F, the lattice $\mathbb{L}(T(\Omega/F)|F^*)$ of all subgroups of $T(\Omega/F)$ lying over F^*, the lattice $\overline{\mathbb{L}}(\Gamma)$ of all closed subgroups of Γ, and the lattice $\mathbb{L}(Z^1(\Gamma, A))$ of all subgroups of $Z^1(\Gamma, A)$ are related as shown in the commutative diagram below:

$$\begin{array}{ccc} \mathbb{L}(T(\Omega/F)|F^*) & \rightleftarrows & \mathbb{I}(\Omega/F) \\ \downarrow & & \downarrow \\ \mathbb{L}(Z^1(\Gamma, A)) & \rightleftarrows & \overline{\mathbb{L}}(\Gamma) \end{array}$$

where the left vertical arrow is the lattice isomorphism induced by ψ, the right vertical arrow is the canonical lattice anti-isomorphism $E \mapsto \Gamma_E := \mathrm{Gal}(\Omega/E)$ (with inverse $\Delta \mapsto E^{\Delta}$) given by the Fundamental Theorem of Infinite Galois Theory, the horizontal top arrows are the sup-semilattice morphism $\mathbb{G} \mapsto F(\mathbb{G})$ and the inf-semilattice morphism $E \mapsto T(E/F)$, while the horizontal bottom arrows are the sup-semilattice anti-morphism $G \mapsto G^{\perp}$ and the inf-semilattice anti-morphism $\Delta \mapsto \Delta^{\perp}$ defined in [4]. The commutativity of the diagram above follows at once from [2, Theorem 1.8] or [3, Theorem 15.1.7]. The next result is essentially a reformulation of the corresponding results from [2] or [3] involving the lattices and the maps above.

Proposition 2.1. *Let E be an intermediate field of the given Galois extension Ω/F, let $\Gamma_E = \mathrm{Gal}(\Omega/E)$, let $A = \mu(\Omega)$, let $\mathbb{G} \in \mathbb{L}(T(\Omega/F)|F^*)$, and let $G = \psi(\mathbb{G})$, where ψ is the canonical group epimorphism $\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A)$ defined above. Then, the following statements hold.*

(1) *The extension E/F is $\mathbb{G}$-radical if and only if the subgroup Γ_E of Γ is G-radical. In particular, E/F is a radical extension (resp. a simple radical extension) if and only if Γ_E is a radical group (resp. a simple radical group) of Γ.*
(2) *The extension E/F is $\mathbb{G}$-Kneser if and only if the subgroup Γ_E of Γ is G-Kneser. In particular, E/F is a Kneser extension if and only if Γ_E is a Kneser group of Γ.*
(3) *The extension $F(\mathbb{G})/F$ is $\mathbb{G}$-Kneser if and only if G is a Kneser group of $Z^1(\Gamma, A)$.*
(4) *The extension E/F is $\mathbb{G}$-Cogalois if and only if the subgroup Γ_E of Γ is G-Cogalois. In this case, G is the unique Cogalois group of $Z^1(\Gamma, A)$ for which $\Gamma_E = G^{\perp}$.*
(5) *The extension $F(\mathbb{G})/F$ is $\mathbb{G}$-Cogalois if and only if G is a Cogalois group of $Z^1(\Gamma, A)$.*
(6) *The extension E/F is Cogalois if and only if the subgroup Γ_E of Γ is strongly Cogalois.*

Proof. (1) is a reformulation of [2, Theorem 1.8] or [3, Theorem 15.1.7], (2) is a reformulation of [2, Corollary 1.10 (1)] or [3, Corollary 15.1.8 (1)], and (4) is a reformulation of [2, Corollary 1.10 (2)] or [3, Corollary 15.1.8 (2)]. The uniqueness of G is assured by [4, Corollary 2.12].

(6) Write $\mathbb{H} = T(E/F)$ and $H = \psi(\mathbb{H}) = \Gamma_E^{\perp}$. By (2), the extension E/F is Cogalois, i.e., $\mathbb{H}$-Kneser, if and only if $\Gamma_E = H^{\perp} = \Gamma_E^{\perp\perp}$ and $\Gamma_E^{\perp}$ is a Kneser group of $Z^1(\Gamma, A)$. By Lemma 0.2, this means precisely that Γ_E is strongly Cogalois. □

Corollary 2.2. *Let Ω/F be a Galois extension, $\Gamma := \mathrm{Gal}(\Omega/F)$, $A := \mu(\Omega)$, and let $F^* \leqslant \mathbb{G} \leqslant T(\Omega/F)$ be such that $E := F(\mathbb{G})$ is a $\mathbb{G}$-Cogalois extension of F. If $G := \psi(\mathbb{G}) \leqslant Z^1(\Gamma, A)$, then the following assertions are equivalent.*

(1) *G is a Γ-submodule of $Z^1(\Gamma, A)$, i.e., it is stable under the action of Γ.*
(2) *E/F is a Galois extension.*
(3) *$\sigma x \in E$ for all $\sigma \in \Gamma$ and $x \in \mathbb{G}$.*

Proof. First, observe that $\Gamma_E := \mathrm{Gal}(\Omega/E) = G^{\perp}$ by Proposition 2.1(4). Now, by [4, Corollary 2.14], G is a Γ-submodule of $Z^1(\Gamma, A) \Longleftrightarrow G^{\perp} \lhd \Gamma \Longleftrightarrow E/F$ is a Galois extension. $\square$

Proposition 2.3. *Let E/F be an arbitrary separable algebraic extension, let $\Omega := \widetilde{F}^{\mathrm{sep}}$, $\Gamma = \mathrm{Gal}(\Omega/F)$, $A = \mu(\Omega)$, and let $n \in \mathbb{N}$ be relatively prime with the characteristic exponent of F.*

Then, the extension E/F is a classical n-Kummer extension (resp. a generalized n-Kummer extension, an n-Kummer extension with few roots of unity, an n-quasi-Kummer extension) if and only if there exists a unique classical n-Kummer group (resp. a generalized n-Kummer group, an n-Kummer group with few cocycles, an n-quasi-Kummer group) G, $G \leqslant Z^1(\Gamma, A)$, such that $\Gamma_E := \mathrm{Gal}(\Omega/E) = G^{\perp}$.

Proof. We may assume that $E \subseteq \Omega$. If E/F is a classical n-Kummer extension, then there exists $\mathbb{G} \in \mathbb{L}(T(E/F)|F^*)$ with $E = F(\mathbb{G})$, $\mathbb{G}^n \subseteq F^*$, and $A[n] = \mu_n(\Omega) \subseteq \mu_n(F) \subseteq A^{\Gamma}$. Let $G = \psi(\mathbb{G})$. Then $nG = \{0\}$, and so G is a classical n-Kummer group. By Proposition 2.1(1), $\Gamma_E = G^{\perp}$. By Proposition 1.2, G is Cogalois, so its uniqueness follows from Proposition 2.1(4).

Conversely, assume that there exists a classical n-Kummer group G of $Z^1(\Gamma, A)$ such that $\mathrm{Gal}(\Omega/E) = G^{\perp}$. If we write $\mathbb{G} = \psi^{-1}(G)$, then $E = F(\mathbb{G})$ by Proposition 2.1 (1). Since clearly $\mathbb{G}^n = 1$ and $\mu_n(\Omega) = A[n] \subseteq A^{\Gamma} \subseteq F$, we deduce that E/F is a classical n-Kummer extension.

The cases of generalized n-Kummer extensions, n-Kummer extensions with few roots of unity, and n-quasi-Kummer extensions follow in the same manner as above from the following facts: $A[n] = \mu_n(\Omega)$ (in particular $A[2] = \{-1, 1\}$) and $A^{\mathrm{Gal}(\Omega/L)} = \{x \in \mu(\Omega) \mid \sigma x = x, \forall \sigma \in \mathrm{Gal}(\Omega/L)\} = \mu(L)$ for any intermediate field L of the given Galois extension Ω/F. $\square$

3. Field Theoretic via Abstract Cogalois Theory

The results of the previous section permit us to retrieve easily most of the basic results of (Field Theoretic) Cogalois Theory [3] from the corresponding results of Abstract Cogalois Theory established in [4]. We will illustrate this by presenting only three of them.

Theorem 3.1 (*The Infinite Kneser Criterion [5, Theorem 2.1] or [3, Theorem 11.1.5]*). *Let E/F be an arbitrary separable $\mathbb{G}$-radical extension. For any $n \in \mathbb{N}$, let $\zeta_n \in \Omega := \widetilde{F}^{\mathrm{sep}}$ denote a primitive n-th root of unity. Then, the following assertions are equivalent.*

(1) *E/F is a $\mathbb{G}$-Kneser extension.*
(2) *$\zeta_p \in \mathbb{G} \Longrightarrow \zeta_p \in F$ for every odd prime p, and $1 \pm \zeta_4 \in \mathbb{G} \Longrightarrow \zeta_4 \in F$.*

Proof. We may assume that $E \subseteq \Omega$. Set $\Gamma := \mathrm{Gal}(\Omega/F)$, $A := \mu(\Omega)$, and let $\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A)$ be the epimorphism defined at the beginning of Section 2. Then $A^{\Gamma} = \mu(F)$ and

$$\mathcal{P}(\Gamma, A) = \{p \mid p \text{ odd prime or 4 such that } \zeta_p \notin F\}.$$

By assumption, $E = F(\mathbb{G})$, with $F^* \leqslant \mathbb{G} \leqslant T(\Omega/F)$. Setting $G := \psi(\mathbb{G}) \leqslant Z^1(\Gamma, A)$ we have $\Gamma_E = G^{\perp}$ by Proposition 2.1(1). So, by Proposition 2.1(3), the extension E/F is $\mathbb{G}$-Kneser if and only if G is a Kneser group of $Z^1(\Gamma, A)$.

For every odd prime $p \neq \mathrm{Char}(F)$, $\varepsilon_p := \psi(\zeta_p) \in Z^1(\Gamma, A)$ is the coboundary assigning to any $\sigma \in \Gamma$ the p-th root of unity $(\sigma\zeta_p)\zeta_p^{-1} \in A[p]$. Obviously, $\varepsilon_p \in G \Longleftrightarrow \zeta_p \in \mathbb{G}$. Observe that if $p = \mathrm{Char}(F) > 2$, then $\zeta_p \in A[p] = \{1\} \subseteq A^{\Gamma}$.

Assume that $\mathrm{Char}(F) \neq 2$. Since $1-\zeta_4 \in T(\Omega/F)$, we can consider the continuous cocycle $\psi(1-\zeta_4) \in Z^1(\Gamma, A)$, which by definition works as follows: $\psi(1 - \zeta_4)(\sigma) = \sigma(1 - \zeta_4) \cdot (1 - \zeta_4)^{-1} = (1 - \sigma\zeta_4) \cdot (1 - \zeta_4)^{-1}, \forall \sigma \in \Gamma$.

Since for any $\sigma \in \Gamma$, we have either $\sigma\zeta_4 = \zeta_4$ or $\sigma\zeta_4 = -\zeta_4$, we deduce that $\psi(1-\zeta_4)(\sigma) = \zeta_4$ if $\sigma\zeta_4 = -\zeta_4$, and $\psi(1-\zeta_4)(\sigma) = 1$ if $\sigma\zeta_4 = \zeta_4$. Thus, $\psi(1-\zeta_4)$ is nothing else than the multiplicative version of the cocycle ε'_4 defined in [4]. A simple calculation shows that $\psi(1+\zeta_4) = (\psi(1-\zeta_4))^{-1}$ in the multiplicative group $Z^1(\Gamma, A)$, so $\varepsilon'_4 \in G \iff 1 \pm \zeta_4 \in \mathbb{G}$. Observe that if $\mathrm{Char}(F) = 2$, then $\zeta_4 \in A[4] = \{1\} \subseteq A^\Gamma$. To finish the proof it remains to apply Proposition 2.1(3) and the Abstract Kneser Criterion [4, Theorem 1.20]. □

Corollary 3.2. *Let E/F be a separable $\mathbb{G}$-radical extension (i.e., $E = F(\mathbb{G})$ for some $F^* \leqslant \mathbb{G} \leqslant T(E/F)$), which is not $\mathbb{G}$-Kneser. Assume that the extension E/F is minimal with respect to the property not being $\mathbb{G}$-Kneser, that is, for any proper subgroup $\mathbb{G}'$ of $\mathbb{G}$, the extension $F(\mathbb{G}')/F$ is $\mathbb{G}'$-Kneser. Then, the extension E/F is cyclic having either the form $E = F(\zeta_p)$ with $p \neq \mathrm{Char}(F)$ an odd prime number and $\zeta_p \notin F$, or the form $F(\zeta_4)$ with $\mathrm{Char}(F) \neq 2$ and $\zeta_4 \notin F$.*

Proof. With $\Omega = \widetilde{F}^{\mathrm{sep}}$, Γ, and A as above, using the epimorphism $\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A)$ as well as Proposition 2.1, we deduce that $G = \psi(\mathbb{G})$ is a minimal non-Kneser group of $Z^1(\Gamma, A)$. According to [4, Lemma 1.18], it follows that either $G = \langle\varepsilon_p\rangle \cong \mathbb{Z}/p\mathbb{Z}$ for some odd prime number $p \neq \mathrm{Char}(F)$ such that $\zeta_p \notin F$, or $G = \langle\varepsilon'_4\rangle \cong \mathbb{Z}/4\mathbb{Z}$, with $\mathrm{Char}(F) \neq 2$ and $\zeta_4 \notin F$. Consequently, $\mathbb{G} = F^*\langle\zeta_p\rangle$ in the former case and $\mathbb{G} = F^*\langle 1+\zeta_4\rangle$ in the latter one. The result now follows. □

Remark 3.3. The inverse implication in Corollary 3.2 may fail. Indeed, $F(\zeta_4)/F$ is $F^*\langle\zeta_4\rangle$-Cogalois, and so Kneser, whenever $\mathrm{Char}(F) \neq 2$ and $\zeta_4 \notin F$. Also, for every prime $p > 2$, if the characteristic exponent of F is prime with $p(p-1)$, $\zeta_p \notin F$, and $\zeta_{p-1} \in F$, then there exists $\theta \in E := F(\zeta_p)$ such that $E = F(\theta)$ and $\theta^{p-1} \in F$; therefore E/F is $F^*\langle\theta\rangle$-Cogalois, and so Kneser.

Theorem 3.4 (*The General Purity Criterion [1, Theorem 2.3] or [3, Theorem 12.1.4]*). *The following assertions are equivalent for an arbitrary separable $\mathbb{G}$-radical extension E/F.*

(1) *E/F is $\mathbb{G}$-Cogalois.*
(2) *E/F is $\mathcal{P}_\mathbb{G}$-pure, i.e., $\zeta_p \in E \Longrightarrow \zeta_p \in E$ for every $p \in \mathcal{P}_\mathbb{G} := \mathcal{P} \cap \mathcal{O}_{\mathbb{G}/F^*}$.*

Proof. We may assume that $E \subseteq \Omega := \widetilde{F}^{\mathrm{sep}}$. Set $\Gamma := \mathrm{Gal}(\Omega/F)$ and $A := \mu(\Omega)$. Since E/F is a $\mathbb{G}$-radical extension, we have $E = F(\mathbb{G})$ with $F^* \leqslant \mathbb{G} \leqslant T(\Omega/F)$. If $G := \psi(\mathbb{G}) \leqslant Z^1(\Gamma, A)$, then $\Gamma_E := \mathrm{Gal}(\Omega/E) = G^\perp$ by Proposition 2.1(1), so E/F is $\mathbb{G}$-Cogalois if and only if G is a Cogalois subgroup of $Z^1(\Gamma, A)$ by Proposition 2.1(5). Since for any $p \in \mathcal{P}_\mathbb{G}$ we have $A^\Gamma[p] = \mu_p(F)$ and $A^{G^\perp}[p] = \mu_p(E)$, we deduce that the $\mathcal{P}_\mathbb{G}$-purity of the extension E/F is equivalent to the quasi-$\mathcal{P}_G$-purity of the embedding $A^\Gamma \leqslant A^{G^\perp}$. The result now follows by applying [4, Theorem 2.5]. □

Theorem 3.5 (*[5, Theorem 3.12] or [3, Theorem 12.1.10]*). *If E/F is a separable algebraic extension which is simultaneously $\mathbb{G}$-Cogalois and $\mathbb{H}$-Cogalois, then $\mathbb{G} = \mathbb{H}$.*

Proof. Apply [4, Corollary 2.12] and Proposition 2.1(4). □

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